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Is the Cabibbo Angle a Function of the Weinberg Mixing Parameter?

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The (u, c) quarks and (d, s) quarks are required to have mass matrices of a certain form. To achieve these mass matrices appropriate Lagrangians are assumed. The u quark is coupled to the standard Higgs scalar ϕ_L. The c quark has a γ_5 coupling with ϕ_L and ϕ_R, where ϕ_R is the Higgs scalar corresponding to the left–right model. The u quark has no γ_5 coupling. Both u, c quarks have a Yukawa coupling with a Higgs multiplet. Exactly similar Lagrangians are chosen for the d, s qurks. Using these mass matrices, the Cabibbo angle is found to be 13° 11′. The ratio m_C/m_S is shown to be approximately 3.1 with the help of the Weinberg mixing parameter. The mixing angles θ_2 and θ_1 determine the Cabibbo angle. The ratio $\tan\theta_2/\tan\theta_1$ is shown to be a function of the Weinberg mixing parameter.

1. INTRODUCTION

An understanding of fermion masses and mixing angles still eludes us. The masses and mixing angles are probably clues to extend the standard model. Any scheme that can account for the observed quark and lepton masses and mixing angles should be welcome. The first generation of fermions are the electron e, its neutrino ν_e; and the u and d quarks. The second generation consists of the muon μ, its neutrino ν_μ, and c and s quarks. The third generation has also been observed.

The mixing between generations manifests itself in the system of quark charged weak currents. By convention, the mixing is assigned to the $Q = -1/3$ quarks by

$$J^{\mu}{}_{ch}(\text{quark}) = u'_{L\alpha}\gamma^{\mu}d'_{L\alpha} = u_L{}'\gamma^{\mu}U_L(u)U_L^{+}(d)\, d_{L\alpha} = u'_{L\alpha}\gamma^{\mu}d''_L \quad (1.1)$$

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where

$$d''_{L,\alpha} = V_{\alpha,\beta}\, d_{L,\beta} \qquad (\alpha, \beta = 1, \ldots, n) \tag{1.2}$$

and

$$V = U_L(u) U_L^{+}(d) \tag{1.3}$$

Thus the $Q = -1/3$ quark states participating in the transitions of the charged weak current are linear combinations of mass eigenstates. The quark mixing matrix V, being the product of two unitary matrices, is itself unitary. The standard model does not predict the content of V. Its matrix elements are extracted experimentally. For the two-generation case V is called the Cabibbo matrix. For three generations it is called the Kobayashi–Maskawa matrix. For n generations V is an $n \times n$ unitary matrix. It is characterized by $n(n - 1)/2$ angles and $n(n + 1)/2$ phases. Not all the phases have physical significance because $2n - 1$ of them can be removed by "quark rephasing."

For two generations, there are no complex phases. The only parameter is commonly taken to be the Cabbibo angle θ_C and we write

$$V = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \tag{1.4}$$

Within the two-generation approximation, weak interaction decay data imply the numerical value $\sin\theta_C \cong 0.228$ or $\theta_C \cong 13^\circ\, 11'$.

The aim of this paper is to show that the Cabibbo angle is related to the Weinberg mixing parameter; the scope of this work is limited to the two generations of quarks. In Section 2 we derive a general formula for the Cabibbo angle. Sections 3 and 4 contain mass matrices for u, c and d, s quarks. In Section 5 the Cabibbo angle is computed. This section also contains a brief discussion of our result.

2. MASS MATRICES AND THE CABIBBO ANGLE

Let the total number of Higgs multiplets that couple to quarks be at most two, so that the electroweak group is $SU(2)_L \times SU(2)_R \times U(1)$. Let there be two generations of quarks. We allow a phase transformation among various generations of fermions such that

$$f_i = e^{i\delta} f_i \tag{2.1}$$

where i is the generation number and δ is an arbitrary phase factor. If we confine consideration only to quarks, Eq. (2.1) shows that each generation of quarks transforms into itself times a phase factor. All the Higgs fields must have well-defined properties under the phase symmetry.

$$\phi_i = e^{i\chi_i}\phi_i \tag{2.2}$$

where χ_i is also arbitrary. Yukawa couplings are allowed if they are invariant under the phase symmetry. We now construct the mass matrices for quarks on the above basis of permutation symmetry and phase transformation for two generations of quarks only.

The above requirements lead surprisingly to a unique nontrivial form of the mass matrix for two generations. For *u*, *c* quarks we have

$$M_1 = \begin{pmatrix} 0 & i\,(m_u m_c)^{1/2} \\ -i\,(m_u m_c)^{1/2} & m_c - m_u \end{pmatrix} \tag{2.3}$$

where m_u and m_c are the constituent masses of the *u* and *c* quarks, respectively. The matrix $M_1M_1^+$ has eigen values m_u^2 and m_c^2, and $M_1M_1^+$ has the following eigenvectors:

$$\frac{1}{(m_u + m_c)^{1/2}}\begin{pmatrix} m_c^{1/2} \\ i m_u^{1/2} \end{pmatrix} \quad \text{and} \quad \frac{1}{(m_u + m_c)^{1/2}}\begin{pmatrix} m_u^{1/2} \\ -i m_c^{1/2} \end{pmatrix}$$

The unitary matrix U_c that diagonalizes the $M_1M_1^+$ mass matrix is given by

$$U_c = \frac{1}{(m_u + m_c)^{1/2}}\begin{pmatrix} m_c^{1/2} & i m_u^{1/2} \\ m_u^{1/2} & -i m_c^{1/2} \end{pmatrix} \tag{2.4}$$

In an exactly similar fashion the mass matrix for *d*, *s* quarks is given by

$$M_2 = \begin{pmatrix} 0 & i\,(m_d m_s)^{1/2} \\ -i\,(m_d m_s)^{1/2} & m_s - m_d \end{pmatrix} \tag{2.5}$$

and $M_2M_2^+$ is diagonalized by the unitary matrix U_s, where

$$U_s = \frac{1}{(m_s + m_d)^{1/2}}\begin{pmatrix} m_s^{1/2} & i m_d^{1/2} \\ m_d^{1/2} & -i m_s^{1/2} \end{pmatrix} \tag{2.6}$$

By definition, $V(\theta_c) = U_c U_s^+$, and

$$V(\theta_c) = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} = \begin{pmatrix} \cos(\theta_2 - \theta_1) & \sin(\theta_2 - \theta_1) \\ -\sin(\theta_2 - \theta_1) & \cos(\theta_2 - \theta_1) \end{pmatrix} \tag{2.7}$$

In the above

$$\theta_C = \theta_2 - \theta_1 = (\tan^{-1}(m_d/m_s)^{1/2} \quad -\tan^{-1}(m_u/m_c)^{1/2}) \tag{2.8}$$

3. THE (*u*, *c*) MASS MATRIX

Let the Higgs sector consist of the multiplet[(1)] $\phi(1/2, 1/2^*, 0)$ and $\tilde{\phi} = \tau_2\phi\,{}^*\tau_2(1/2, 1/2^*, 0)$, such that

$$\langle \phi \rangle = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa' \end{pmatrix} \tag{3.1}$$

where κ and κ' are real. In addition to the above multiplet, the Higgs sector has the Higgs scalar ϕ_L corresponding to the standard model, and another Higgs scalar ϕ_R corresponding to the left–right model, with

$$\langle \phi_L \rangle = V_L \tag{3.2}$$

$$\langle \phi_R \rangle = V_R \tag{3.3}$$

Then, let

$$-L_1 = h_1 \overline{Q}_L \phi Q_R + h_2 \overline{Q}_L \tilde{\phi} Q_R + \text{H.c.} \tag{3.4}$$

where

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \tag{3.5}$$

From the above Lagrangian we note that the masses of u and d quarks are given by

$$m_u = m_1 = h_1 k + h_2 k' \tag{3.6}$$

and

$$m_d = m_3 = h_1 k' + h_2 k \tag{3.7}$$

The Yukawa coupling constants h_1 and h_2 are real. In an exactly similar way we assume that ϕ is also coupled to c and s quarks such that

$$-L_2 = h_3 \overline{Q}_L \phi Q_R + h_4 \overline{Q}_L \tilde{\phi} Q_R + \text{H.c.} \tag{3.8}$$

where

$$Q_L = \begin{pmatrix} c_L \\ s_L \end{pmatrix} \qquad \text{and} \qquad Q_R = \begin{pmatrix} c_R \\ s_R \end{pmatrix} \tag{3.9}$$

At this stage, the masses of c and s are, respectively,

$$m_2 = h_3 k + h_4 k' \tag{3.10}$$

$$m_4 = h_4 k + h_3 k' \tag{3.11}$$

The coupling constants h_3 and h_4 are real.

In addition to the above Lagrangians, the u and c quarks are also coupled[(2)] to ϕ_L. Before examining this, let us note an important item. Given a Dirac field ψ, the Hermitian scalar and pseudoscalar $\overline{\psi}\psi$ and $i\overline{\psi}\gamma_5\psi$ have opposite CP and T transformation properties. (In this respect they are unlike the vector and axial vector $\bar{\psi}\gamma_\lambda\psi$, $\bar{\psi}\gamma_\lambda\gamma_5\psi$). This is the key to the *CP* violation by a Higgs field. The simplest model uses a Higgs field ϕ_L and a Lagrangian containing

$$m\bar{\psi}\psi + ia\bar{\psi}\gamma_5\psi\phi_L \tag{3.12}$$

where a is a real coupling constant. This conserves *CP* if ϕ_L is assigned $CP = -1$. But if spontaneous symmetry breaking gives ϕ_L a nonzero VEV, V_L, (3.12), may be written

$$(m^2 + a^2V_L^2)^{1/2}\bar{\psi}'\psi' + a\bar{\psi}'(\sin\alpha + i\gamma_5\cos\alpha)\psi'\phi_L' \tag{3.13}$$

where

$$\phi_L = V_L + \phi_L' \qquad \text{and} \qquad \psi = \exp(-\tfrac{1}{2}i\gamma_5\alpha)\psi' \tag{3.14}$$

and

$$\tan\alpha = aV_L/m \tag{3.15}$$

Vector or axial-vector interactions are unaffected by the transformation from ψ to ψ'. The *CP* violation is now caused by the exchange of ϕ_L' particles.[3]

To implement the above scheme in the case of u, c quarks the following Lagrangian is chosen:

$$\begin{aligned}-L_3 = {} & m_1\bar{u}u + m_2\bar{c}c - a_1\bar{u}u\phi_L + ia_0\bar{u}c\phi_L - ia_0\bar{c}u\phi_L \\ & + ia_L\bar{c}\gamma_5c\phi_L + ia_R\bar{c}\gamma_5c\phi_R\end{aligned} \tag{3.16}$$

where a_1, a_0, a_L, and a_R are real. The first two terms are the contributions of Eqs. (3.6) and (3.10). After spontaneous symmetry breaking and due to the following transformations and restrictions,

$$c = \exp(-i\gamma_5\,\alpha_1/2)\,c' \tag{3.17}$$

$$u = \exp(-i\gamma_5\,\alpha_2/2)\,u' \tag{3.18}$$

and

$$m_1 = a_1 V_L \qquad \text{and} \qquad \alpha_1 + \alpha_2 = 0 \tag{3.19}$$

we can write the Lagrangian L_3 in the following way:

$$\begin{aligned}-L_3 = {} & 0\,\bar{u}'u' + ia_0V_L\bar{u}'c' - ia_0V_L\bar{c}'u' \\ & + [m_2^2 + (a_LV_L + a_RV_R)^2]^{1/2}\bar{c}'c' - (a_1\cos\alpha_1)\bar{u}'u'\phi_L' \\ & + ia_0\bar{u}'c'\phi_L' - ia_0\bar{c}'u'\phi_L' - i(a_1\sin\alpha_1)\bar{u}'\gamma_5u'\phi_L' \\ & + a_L\bar{c}'[i\gamma_5\cos\alpha_1 + \sin\alpha_1]c'\phi_L' \\ & + a_R\bar{c}'[i\gamma_5\cos\alpha_1 + \sin\alpha_1]c'\phi_R'\end{aligned} \tag{3.20}$$

To avoid such constant terms like $\bar{c}'\gamma_5 c'$ in the above Lagrangian we require that

$$\tan \alpha_1 = (a_L V_L + a_R V_R)/m_2 \tag{3.21}$$

In addition, $\alpha_1 + \alpha_2 = 0$ ensures the absence of terms like $\bar{u}'\gamma_5 c' V_L$ and $\bar{u}'\gamma_5 c'\phi_L'$ and their Hermitian conjugates. The choice $m_1 = a_1 V_L$ is required to obtain a mass matrix that has the desired form as in Eq. (2.2). So finally this choice of the Lagrangian along with the constraints leads to the following mass matrix for the u, c quarks:

$$(\bar{u}, \bar{c})\begin{pmatrix} 0 & i a_0 V_L \\ -i a_0 V_L & [m_2^2 + (a_L V_L + a_R V_R)^2]^{1/2} \end{pmatrix}\begin{pmatrix} u \\ c \end{pmatrix} \tag{3.22}$$

The parameters a_0, m_2, a_L, and a_R are free parameters of the model. There are no restrictions on them. Suppose

$$[m_2^2 + (a_L V_L + a_R V_R)^2]^{1/2} = [m_d M_{WL} B_1 (1 - A_1)]^{1/2} - m_1 \tag{3.23}$$

and

$$a_0 V_L = [\{[m_2^2 + (a_L V_L + a_R V_R)^2]^{1/2} + m_1\}\, m_1]^{1/2} \tag{3.24}$$

With this choice the eigenvalues of $M_1 M_1^+$ are

$$m_u^2 = m_1^2 \tag{3.25}$$

$$m_c^2 = m_d M_{WL} B_1 (1 - A_1) \tag{3.26}$$

In obtaining these expressions we have used a result obtained earlier.(2) The constants B_1 and A_1 are given by

$$B_1 = \frac{(g_V/g_A)^4_{ds}}{(g_V/g_A)^4_{uc}} \tag{3.27}$$

$$A_1 = [1 - (g_V/g_A)^4_{uc}]^{1/2} \tag{3.28}$$

Here g_V and g_A are the vector and axial vector coupling constants of the particles indicated by the subscripts with the Z-particle of the standard model. Equation (3.26) along with Eqs. (3.27) and (3.28) for m_c^2 can be written down by just examining a similar expression for m_e^2 derived in ref. 2. Finally, using (3.27) and (3.28) in Eq. (3.26), we have

$$m_c^2 = m_d M_{WL} \frac{(g_V/g_A)^4_{ds}}{(g_V/g_A)^4_{uc}} \left\{1 - \left[1 - \left(\frac{g_V}{g_A}\right)^4_{uc}\right]^{1/2}\right\} \tag{3.29}$$

4. THE (d, s) MASS MATRIX

We follow an exactly similar procedure in the case of d, s quarks. The d, s quarks are coupled to ϕ_L and ϕ_R in the following way:

$$-L_4 = m_3\bar{d}d + m_4\bar{s}s - a_2\bar{d}d\phi_L + ib_0\,\bar{d}s\,\phi_L - ib_0\,\bar{s}d\,\phi_L + ib_L\bar{s}\gamma_5 s\,\phi_L + ib_R\bar{s}\gamma_5 s\,\phi_R \tag{4.1}$$

In the above a_2, b_0, b_L, and b_R are all real. Moreover, m_3 and m_4 are contributions of Eqs. (3.7) and (3.11). In addition,

$$d = \exp(-i\,\gamma_5\,\alpha_3/2)\,d' \tag{4.2}$$

$$s = \exp(-i\gamma_5\alpha_4/2)\,s' \tag{4.3}$$

In addition to the above we require that

$$\alpha_3 + \alpha_4 = 0 \text{ and } m_3 = a_2\,V_L. \tag{4.4}$$

With these conditions after spontaneous symmetry breaking the Lagrangian (4.1) may be written as

$$\begin{aligned}-L_4 = {} & 0\,\bar{d}'d' + ib_0V_L\,\bar{d}'s' - ib_0V_L\,\bar{s}'d' \\ & + [m_4^2 + (b_LV_L + b_RV_R)^2]^{1/2}\,\bar{s}'\,s' - (a_2\cos\alpha_4)\bar{d}'\,d'\,\phi_L' \\ & + ib_0\bar{d}'s'\phi_L' - ib_0\bar{s}'d'\,\phi_L' - i(a_2\sin\alpha_4)\bar{d}'\,\gamma_5\,d'\,\phi_L' \\ & + b_L\bar{s}'[i\gamma_5\cos\alpha_4 + \sin\alpha_4]s'\,\phi_L' \\ & + b_R\bar{s}'[i\gamma_5\cos\alpha_4 + \sin\alpha_4]s'\,\phi_R'\end{aligned} \tag{4.5}$$

Again to avoid $\bar{s}'\gamma_5 s'$ type of terms we require that

$$\tan\alpha_4 = (b_LV_L + b_RV_R)/m_4 \tag{4.6}$$

The mass matrix M_2 for d, s quarks is given by

$$M_2 = \begin{pmatrix} 0 & ib_0V_L \\ -ib_0V_L & [m_4^2 + (b_L\,V_L + b_RV_R)^2]^{1/2} \end{pmatrix} \tag{4.7}$$

The parameters b_0, b_L, b_R, and m_4 are still free.

With the choice

$$[m_4^2 + (b_LV_L + b_RV_R)^2]^{1/2} = [m_u\,M_{WL}B_2(1 - A_2)]^{1/2} - m_3 \tag{4.8}$$

and

$$b_0V_L = [\{[m_4^2 + (b_LV_L + b_RV_R)^2]^{1/2} + m_3\}m_3]^{1/2} \tag{4.9}$$

we have

$$B_2 = \frac{(g_V/g_A)^4_{uc}}{(g_V/g_A)^4_{ds}} \tag{4.10}$$

and

$$A_2 = [1 - (g_V/g_A)^4_{ds}]^{1/2} \tag{4.11}$$

We note that

$$m_d{}^2 = m_3{}^2 \tag{4.12}$$

and

$$m_s{}^2 = m_u\, M_{WL}\, \frac{(g_V/g_A)^4_{uc}}{(g_V/g_A)^4_{ds}} \left\{ 1 - \left[1 - \left(\frac{g_V}{g_A} \right)^4_{ds} \right]^{1/2} \right\} \tag{4.13}$$

The resemblance of this expression to Eq. (3.29) should be noticed. Of course the above expression can also be directly written down by just examining the expression for $m_e{}^2$ in ref. 2. Here u, d play the same role as the neutrino does in the case of $m_e{}^2$.

5. CABBIBO ANGLE AND DISCUSSION

In Eqs. (3.29) and (4.13), g_v and g_A are the vector and axial vector coupling constants of the particles indicated by the subscripts with the Z particle of the standard model. Moreover, m_d and m_u are the constituent masses of down and up quarks. For numerical calculations we assume that

$$m_u \approx m_d = 0.3 \text{ GeV} \tag{5.1}$$

From the standard model prescription we know that

$$\begin{aligned} (g_v/g_A)_d{}^2 &= (g_v/g_A)_s{}^2 = (g_v/g_A)_b{}^2 = (-1 + \tfrac{4}{3}X_L)^2 \\ (g_v/g_A)_u{}^2 &= (g_v/g_A)_c{}^2 = (g_v/g_A)_t{}^2 = (-1 + \tfrac{8}{3}X_L)^2 \end{aligned} \tag{5.2}$$

Here $X_L = \sin^2\theta_W$, where θ_W is the Weinberg mixing angle. From (3.29) and (4.13) we observe that $m_c = 1.7$ GeV and $m_s = 0.57$ GeV provided $M_{WL} =$ 80 GeV and $X_L = 0.2254$.(2)

Therefore,

$$\theta_2 = \tan^{-1}(m_d/m_s)^{1/2} = 35°58' \tag{5.3}$$

$$\theta_1 = \tan^{-1}(m_u/m_c)^{1/2} = 22°47', \tag{5.4}$$

and so the Cabibbo angle is

$$\theta_C = \theta_2 - \theta_1 = 13°11' \tag{5.5}$$

and

$$\sin \theta_C = 0.228 \tag{5.6}$$

This agrees pretty well with the experimental value noted in the introduction.

Equations (3.29) and (4.13) can be approximated by

$$2m_c^2 \approx m_d M_{\mathrm{WL}} (g_{\mathrm{v}}/g_{\mathrm{A}})_d^4 \tag{5.7}$$

and

$$2m_s^2 \approx m_u M_{\mathrm{WL}} (g_{\mathrm{v}}/g_{\mathrm{A}})_u^4 \tag{5.8}$$

The above expressions yield the ratio

$$\frac{m_c}{m_s} \approx \frac{(g_{\mathrm{v}}/g_{\mathrm{A}})_d^2}{(g_{\mathrm{v}}/g_{\mathrm{A}})_u^2} \approx 3.1 \tag{5.9}$$

This ratio depends only on the Weinberg mixing parameter. This ratio is well known from many experiments. Here the masses are constituent masses. Equation (5.9) can be compared with experiment. The important point is that Eq. (5.9) is a prediction based on the standard model.

From the approximate expressions for m_c^2 and m_s^2 it also follows that

$$\tan \theta_2 \approx \left[\frac{2^{1/2} m_d}{(m_u M_{\mathrm{WL}})^{1/2} \, (g_{\mathrm{v}}/g_{\mathrm{A}})_u^2} \right]^{1/2} \approx \left[\frac{(2m_u)^{1/2}}{(M_{\mathrm{WL}})^{1/2} \, (g_{\mathrm{v}}/g_{\mathrm{A}})_u^2} \right]^{1/2} \tag{5.10}$$

and

$$\tan \theta_1 \approx \left[\frac{2^{1/2} m_u}{(m_d M_{\mathrm{WL}})^{1/2} \, (g_{\mathrm{v}}/g_{\mathrm{A}})_d^2} \right]^{1/2} \approx \left[\frac{(2m_u)^{1/2}}{(M_{\mathrm{WL}})^{1/2} \, (g_{\mathrm{v}}/g_{\mathrm{A}})_d^2} \right]^{1/2} \tag{5.11}$$

From the above we notice that

$$\frac{\tan \theta_2}{\tan \theta_1} \approx \left[\frac{(g_{\mathrm{v}}/g_{\mathrm{A}})_d^2}{(g_{\mathrm{v}}/g_{\mathrm{A}})_u^2} \right]^{1/2} \tag{5.12}$$

In this paper Lagrangians L_3 and L_4 are chosen in such a way that the mass matrices of (u, c) and (d, s) quarks have a desired form. The Lagrangians contain free parameters which are chosen so as to lead to known expressions for m_c^2 and m_s^2. However, these free parameters of the Lagrangians can be experimentally determined once the Higgs multiplets are discovered. But the important point is that now the content of the mixing matrix is determined by the standard model or extensions thereof. Much experimental data exist on the four-quark model. These data can be used to find the ratio m_c/m_s and through it the Weinberg mixing parameter can be evaluated. If it agrees with the experiment approximately, then the contents of this paper are on a solid foundation. For the first time relations like (5.9) and (5.12) are obtained where the quark mass ratio and mixing angles appear as functions of the Weinberg mixing parameter.

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REFERENCES

1. R. N. Mohapatra and G. Sanjanovic (1981). *Phys. Rev.* **D 23**, 165.
2. Cvavb. Chandra Raju (1997). *Int. J. Theor. Phys.* **36**, 2937.
3. R. N. Mohapatra (1972). *Phys. Rev. D* **6**, 2023; R. N. Mohapatra and J. C. Pati (1975). *Phys. Rev. D* **11**, 566; A. Pais (1973). *Phys. Rev. D* **8**, 625; D. Bailin and A. Love (1974). *Nucl. Phys. B* **69**, 142; see also J. C. Taylor, *Gauge Theories of Weak Interactions* (Cambridge University Press, Cambridge, 1976).